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STRESSED-CONSTRUCTION RIGIDITY CHARACTERISTICS

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It has been shown [1-5] that the averaging method must be applied directly to the initial body in order to incorporate correctly the preliminary (initial) stresses if the body is inhomogeneous. In [1-5], this fact is merely noted and illustrated on examples. Methods of realizing the theoretical results are not considered in [1-5] for particular cases, except in [4], which deals with layered media.

Here a method of incorporating the initial stresses in application to finite-dimensional constructions (trusses, frames, and so on) is given. Working formulas are derived that reduce the treatment to a system of linear equations. One incorporates the preliminary (initial) stresses, which are caused in the main by the weight, as an important feature in considering the stability and dynamic properties.

1. Formulation and Information on A:≶raging for Inhomogeneous Stressed Media.

Consider an inhomogeneous elastic body having a periodic structure, which has periodicity cell PC, P_ε, which is subjected to forces F, which cause displacements $\mathbf{v}^\varepsilon$ and stresses $\sigma_{ij}{}^\varepsilon(0)(\mathbf{v}^\varepsilon)$, which are called the preliminary or initial ones. In addition to the above, the body may be subject to additional displacements $\mathbf{u}^\varepsilon$. The general description of a body containing initial stresses has been considered in [6], and the following have been derived (Fig. 1) for describing the basic (initial) state:

$$L_\varepsilon(0)\mathrm{v}^\varepsilon = \mathrm{F} \text{ в } Q_\varepsilon,\ \sigma^\varepsilon_{ij}(0)(\mathrm{v}^\varepsilon)n_j = 0 \text{ on } \Gamma_\varepsilon,\ \mathrm{v}^\varepsilon = 0 \text{ on } \Gamma_2; \tag{1.1}$$

and for determining the additional displacements:

$$L_\varepsilon(\sigma)\mathrm{u}^\varepsilon = \rho \mathrm{u}^\varepsilon_{,tt} \text{ in } Q_\varepsilon,\ \sigma^\varepsilon_{ij}(\sigma)(\mathrm{u}^\varepsilon)n_j = 0 \text{ on } \Gamma_\varepsilon,\ \mathrm{u}^\varepsilon = 0 \text{ on } \Gamma_2. \tag{1.2}$$

Here $L_\varepsilon(\sigma)\mathrm{v} = [(c_{ijkl}(\mathrm{x}/\varepsilon) + \sigma_{ik}{}^\varepsilon(\sigma)(\mathrm{v}^\varepsilon)\delta^j{}_l)\mathrm{v}_{k,l}]_j$ is the elasticity-theory operator that incorporates the initial stresses [6], while $L_\varepsilon(0)\mathbf{u} = [c_{ijkl}(\mathrm{x}/\varepsilon)\mathrm{u}_{k,l}]_j$ is that operator without the initial stresses,

$$\sigma^\varepsilon_{ij}(\sigma)(\mathrm{u}) = (c_{ijkl}(\mathrm{x}/\varepsilon) + \sigma^\varepsilon_{jl}(\mathrm{x})\delta^i_k)u_{k,l};$$

with $\sigma^\varepsilon_{ij}(0)(\mathrm{u}) = c_{ijkl}(\mathrm{x}/\varepsilon)u_{k,l}$ the stresses, $\sigma^\varepsilon_{ij} = \sigma^\varepsilon_{ij}(0)(\mathrm{v}^\varepsilon)$; $c_{ijkl}(\mathrm{x}/\varepsilon)$ the elastic constants, $\rho(\mathrm{x}/\varepsilon)$ the density (these functions are periodic in x with PC P_ε), and $\delta_j{}^i$ is the Kronecker delta. Figure 1 shows the region Q and its boundaries Γ_ε and Γ_2, with n the normal to ∂Q_ε.

From [7-13], this body with its periodic structure for $e \to 0$ can be replaced by a homogeneous body similar to it in mechanical behavior. Correspondingly, the solutions to Eqs. (1.1) and (1.2) may be approximated by ones of the following form [1-5]:

$$\hat{L}(0)\mathrm{v} = \mu\mathbf{F} \text{ in } Q,\ \hat{\sigma}_{ij}(0)n_j = 0 \text{ on } \Gamma_1,\ \mathrm{v} = 0 \text{ on } \Gamma_2; \tag{1.3}$$

$$\hat{L}(\sigma)\mathbf{u} = \mu\rho\mathbf{u}_{,tt} \text{ in } Q,\ \hat{\sigma}_{ij}(\sigma)n_j = 0 \text{ on } \Gamma_1,\ \mathbf{u} = 0 \text{ on } \Gamma_2, \tag{1.4}$$

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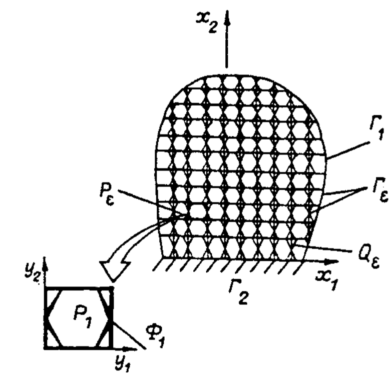

Fig. 1

in which $(\hat{L}(\sigma)\mathbf{u})_i = (a_{ijkl}(\sigma)u_{k,l})_j$ (for $\sigma_{ij}^\varepsilon = 0$ we get an averaged treatment for an unstressed body), with $\sigma_{ij}(\sigma)(u) = a_{ijkl}(\sigma)u_{kl}$ the averaged stresses, $\hat{\sigma}_{ij} = \hat{\sigma}_{ij}(0)(v)$; $\langle\cdot\rangle = (\text{mes } P_\varepsilon)^{-1}\int_{\Phi_\varepsilon} \cdot\, dx = (\text{mes } P_1)^{-1}\int_{\Phi_1} \cdot\, dy$, $y = x/\varepsilon$ mean over the PC (Fig. 1), and μ the volume content of the material ($\mu = 1$ for a monolithic body and $0 < \mu < 1$ for a porous one): $\mu = \text{mes } \Phi_1/\text{mes } P_1$.

From [10],

$$\langle\sigma_{ij}^\varepsilon\rangle = \hat{\sigma}_{ij}(0)(v). \tag{1.5}$$

In the general case (see [1-5, 8, 11] and examples there),

$$a_{ijkl}(\sigma) \neq a_{ijkl}(0) + \hat{\sigma}_{jl}(0)(v)\delta^i_k. \tag{1.6}$$

The right-hand side in Eq. (1.6) arises when one uses what is called intermediate averaging, which is carried out as follows: one averages over the inhomogeneous body free from stresses and calculates the averaged stresses in it and then one compiles an operator that should arise [6] in describing a real homogeneous body having those elastic constants and initial stresses. The intermediate averaging leads in particular to a phenomenological approach to an inhomogeneous body. It follows from Eq. (1.6) that intermediate averaging in general gives an incorrect result. Mathematically, this is due to the specific features of the G limit to a sum [14], and from the mechanical viewpoint it is explained by the occurrence of a general state of stress and strain when homogeneous averaged stresses are applied in an inhomogeneous medium.

The averaging procedure thus in general does not preserve the structure of the operator $L_\varepsilon(\sigma)$ from linearized elasticity theory for bodies containing initial stresses.

2. Small Initial Stresses.

Here stressed constructions are considered. There are naturally constraints on the initial (preliminary) stresses, i.e., σ_{ij}^ε will not exceed the strength of the material. In turn, the ultimate strength for a real material is small by comparison with the elastic constants [14], so the case is that of small initial stresses, for which a formula has been derived [1, 5] for the coefficients in the averaged operator:

$$a_{ijkl}(\sigma) = a_{ijkl}(0) + \hat{\sigma}_{jl}\delta^i_k + l_{ijkl}(\sigma^\varepsilon_{mn}, N^{\alpha\beta}) = a_{ijkl}(0) + \hat{\sigma}_{jl}\delta^i_k + \langle\sigma^\varepsilon_{qm}N^{kl}_{p,my}N^{ij}_{p,qy} + \sigma^\varepsilon_{ql}N^{ij}_{k,qy} + \sigma^\varepsilon_{mj}N^{kl}_{i,my}\rangle. \tag{2.1}$$

Note 2.1. Formula (2.1) incorporates the fact that $\langle\sigma^\varepsilon_{jl}\rangle = \hat{\sigma}_{jl}(0)(v)$ (see [10], where the normalization used in [10] in defining the mean should be replaced by the standard one to derive Eq. (1.5)), while the subscript, iy denotes $\partial/\partial y_i$, where $y = x/\varepsilon$ are the local variables.

Formula (2.1) has been derived by expanding the general formula for $a_{ijkl}(\sigma)$ from [1, 5] in powers of the small parameter σ/c (σ and c are the characteristic values of the initial stresses σ_{ij}^ε and the elastic constants c_{ijkl}) and then retaining the linear term. If necessary, one can retain higher-order terms (see details in [1]). The general formula for $a_{ijkl}(\sigma)$ has been given in [1, 5] and is not given here because it is not used directly in the subsequent steps.

The functions $N^{\alpha\beta}$ in Eq. (2.1) are solutions to the cellular problem CP in elasticity theory for a body free from initial stresses (see 7-9, 12]):

$$(c_{ijkl}(y)(N_k^{\alpha\beta} + y_\alpha\delta_\beta^k)_{,ly})_{,jy} = 0 \text{ in PC } \Phi_1 = \varepsilon^{-1}\Phi_\varepsilon \tag{2.2}$$

together with the condition for the periodicity of $N^{\alpha\beta}$ and the normalization condition $\langle N^{\alpha\beta}\rangle = 0$ (the latter can be replaced by any other condition that eliminates displacement of the body as a solid body).

It follows from [7-10] that the initial stresses $\sigma_{ij}^{\varepsilon}$ are related to the averaged initial stresses by

$$\sigma_{ij}^{\varepsilon} = (c_{ij\alpha\beta} + c_{ijpq}N_{p,qy}^{\alpha\beta})v_{\alpha,\beta} = (c_{ij\alpha\beta} + c_{ijpq}N_{p,qy}^{\alpha\beta})J_{\alpha\beta mn}\sigma_{mn} \tag{2.3}$$

($J_{\alpha\beta mn}$ is a tensor inverse to $a_{ijkl}(0)$, the averaged compliance tensor).

Note 2.2. The local stresses $\sigma_{ij}^{\varepsilon}$ satisfy

$$\sigma_{ij,jy}^{\varepsilon} = 0, \tag{2.4}$$

which can be derived by differentiating Eq. (2.3) with respect to the local variables and using Eq. (2.2). The initial-stress tensor is symmetrical, so we integrate by parts to get

$$\int_{\Phi_1} \sigma_{ql}^{\varepsilon} N_{k,qy}^{\alpha\beta} dy = \int_{\Phi_1} \sigma_{lq}^{\varepsilon} N_{k,qy}^{\alpha\beta} dy = -\int_{\Phi_1} \sigma_{lq,qy}^{\varepsilon} N_k^{\alpha\beta} dy + \int_{\partial\Phi_1} \sigma_{lq}^{\varepsilon} n_q^1 N_k^{\alpha\beta} dy = 0$$

(n^1 is the normal to $\partial\Phi_1$). The first integral in the above sum is equal to zero by virtue of Eq. (2.4), while the second (with respect to $\partial\Phi_1$) is so by virtue of the periodicity of $N^{\alpha\beta}$ and $\sigma_{lq}^{\varepsilon}$ and because the vectors for the normals to opposite faces of the PC are opposite in direction. Then Eq. (2.1) becomes

$$a_{ijkl}(\sigma) = a_{ijkl}(0) + \sigma_{jl}\delta_{k}^{i} + \langle\sigma_{qm}^{\varepsilon} N_{p,my}^{kl} N_{p,qy}^{ij}\rangle. \tag{2.5}$$

This formula is the basic one for subsequent studies.

It follows from Eq. (2.2) by virtue of the elastic-constant symmetry [14] that $N^{\alpha\beta}$ are symmetrical with respect to the superscripts, while the unsymmetry in $a_{ijkl}(\sigma)$ is associated only with the $\hat{\sigma}_{ik}\delta_{jl}$ term (the $a_{ijkl}(0)$ have [7-10] the symmetries occurring in the elastic constants). The conclusion applies only to small initial stresses (see some formulas for not very small initial stresses in [4, 5]).

We substitute the last expression from Eq. (2.3) into Eq. (2.5) in place of $\sigma_{ij}^{\varepsilon}$ to get

$$l_{ijkl}(\sigma_{mn}^{\varepsilon}, N^{\alpha\beta}) = \langle\sigma_{qm}^{\varepsilon} N_{p,my}^{kl} N_{p,qy}^{ij}\rangle = \hat{l}_{ijkl\alpha\beta}(N^{\gamma\delta}) J_{\alpha\beta mn}\hat{\sigma}_{mn}, \tag{2.6}$$

in which

$$\hat{l}_{ijkl\alpha\beta}(N^{\gamma\delta}) = \langle c_{qs\varepsilon d} N_{c,dy}^{\alpha\beta} N_{p,sy}^{kl} N_{p,qy}^{ij} + c_{qs\alpha\beta} N_{p,sy}^{kl} N_{p,qy}^{ij}\rangle; \tag{2.7}$$

with summation with respect to the repeating subscripts.

3. Structures Made of Rectilinear of Planar Elements.

We consider a construction of periodic structure and formed of beams, plates, and so on. Such a structure is a particular case of a highly porous framework structure [2, 10, 11]. In that case, Eq. (2.2) can be replaced as proposed in [12, 13] by means of methods from CP theory for the corresponding cellular construction CC formed by a system of beams and/or plates.

We consider Eq. (2.2). It can be considered as a problem on $\mathbf{U}^{\alpha\beta} = \mathbf{N}^{\alpha\beta} + y_\alpha \mathbf{e}_\beta$, where CP theory for beams/plates is [12, 13] formulated naturally in terms of $\mathbf{U}^{\alpha\beta}$ in the sense that the kinematic hypotheses link the displacements of the CC elements to $\mathbf{U}^{\alpha\beta}$, as (2.2) shows. Then we get (2.7) in terms of $\mathbf{U}^{\alpha\beta}$:

$$l_{ijkl}(\sigma^{\varepsilon}_{mn}, \mathbf{N}^{\alpha\beta}) = \langle \sigma^{\varepsilon}_{qm}(U^{kl}_{p,my} - \delta^k_p \delta^l_m)(U^{ij}_{p,qy} - \delta^k_p \delta^l_q)\rangle = \langle \sigma^{\varepsilon}_{qm} U^{kl}_{p,my} U^{ij}_{p,qy} - \sigma^{\varepsilon}_{ql} U^{ij}_{k,qy} - \sigma^{\varepsilon}_{jm} U^{kl}_{i,my} + \sigma^{\varepsilon}_{jl}\delta^i_k \rangle \tag{3.1}$$

and correspondingly

$$\hat{l}_{ijkl\alpha\beta}(\mathbf{N}^{\gamma\delta}) = \langle c_{qlcd}(U^{\alpha\beta}_{c,dy} - \delta^\alpha_c \delta^\beta_d)(U^{kl}_{p,sy} - \delta^k_p \delta^l_s)(U^{ij}_{p,qy} - \delta^i_p \delta^j_q) + c_{qs\alpha\beta}(U^{kl}_{p,sy} - \delta^k_p \delta^l_s)(U^{ij}_{p,qy} - \delta^i_p \delta^j_q)\rangle. \tag{3.2}$$

Expanding the parentheses in Eq. (3.2) does not simplify the formula.

The theory of beams/plates establishes a relation between the displacements of the elements (considered as one-dimensional or two-dimensional objects) and $\mathbf{U}^{\alpha\beta}$ (the displacements of the elements considered as three-dimensional bodies) in the simplest form in the natural coordinate system linked to the elements [15]. We introduce the $\{l, n, \tau\}_I$ coordinate system linked to body I. Vector $\mathbf{n}$ is taken as the vector for the normal for beams and plates, while vector l is the direction vector for beams.

Formulas (3.1) and (3.2) are written as follows in a coordinate system linked to element I (γ_a^i denote the direction $\{l, n, \tau\}_I$ coordinate system relative to the coordinate system $Oy_1y_2y_3$, $i = 1, 2, 3$, $a = l, n, \tau$):

$$l_{ijkl}(\sigma^{\varepsilon}_{mn}, \mathbf{N}^{\alpha\beta}) = \langle \sigma^{\varepsilon}_{qm} U^{kl}_{p,my} U^{ij}_{p,qy} - \gamma^k_a \gamma^l_b \sigma^{\varepsilon}_{qb} U^{ij}_{a,qy} - \gamma^i_a \gamma^j_b \sigma^{\varepsilon}_{bm} U^{kl}_{a,my} + \gamma^i_a \gamma^j_b \gamma^k_\alpha \gamma^l_\beta \sigma^{\varepsilon}_{b\beta} \delta^a_\alpha \rangle; \tag{3.3}$$

$$\hat{l}_{ijkl\alpha\beta}(\mathbf{N}^{\gamma\delta}) = \langle c_{qlcd}(U^{\alpha\beta}_{c,dy} - \delta^\alpha_c \delta^\beta_d)(U^{kl}_{p,sy} - \delta^k_p \delta^l_s)(U^{ij}_{p,qy} - \delta^i_p \delta^j_q) + \gamma^\alpha_a \gamma^\beta_b c_{qlab}(U^{kl}_{p,sy} - \delta^k_p \delta^l_s)(U^{ij}_{p,qy} - \delta^i_p \delta^j_q)\rangle. \tag{3.4}$$

Here the subscripts run through the values n, l, and τ and the superscripts through the values 1, 2, and 3.

The mean over the PC P_1 in the present case is

$$\langle \cdot \rangle = (\mathrm{mes} P_1)^{-1} \sum_{I=1}^{N} \int_{L_I} \cdot \, dy,$$

where N is the number of elements in the CC; the integration is taken over the region L_I occupied by element I. The integrals can be calculated explicitly on the basis of the hypotheses from beam/plate theory.

3.1. Beam Structures (Frames). We write the following formulas for the stresses and strains [15-17] in the coordinate system linked to beam I (the beam material is taken as homogeneous and isotropic, and the beam has a constant cross section:

$$\sigma^{\varepsilon}_{ll} \neq 0, \ \sigma_{ab} = 0 \ \text{ for } \ ab \neq ll,$$
$$\left.\begin{array}{c} U^{\alpha\beta}_{l,l} \neq 0, \ U^{\alpha\beta}_{n,n} = U^{\alpha\beta}_{\tau,\tau} = -\nu U^{\alpha\beta}_{l,l} \\ U^{\alpha\beta}_{a,b} = 0 \ \text{ for } \ a \neq b \end{array}\right\} \ \text{for any} \ \ \alpha, \beta. \tag{3.5}$$

We represent Eqs. (3.3) and (3.4) on the basis of Eqs. (3.5) as

$$l_{ij\alpha\beta}(\sigma^{\varepsilon}_{mn}, \mathbf{N}^{\gamma\delta}) = (\mathrm{mes} P_1)^{-1} \sum_{I=1}^{N} \int_{L_I} \sigma^{\varepsilon}_{ll}[U^{\alpha\beta}_{l,l} U^{ij}_{l,l} - \gamma^\alpha_l \gamma^\beta_l (U^{ij}_{l,l} + U^{\alpha\beta}_{l,l}) + \gamma^i_l \gamma^j_l \gamma^\beta_l \gamma^\alpha_l] \, dl \, dn \, d\tau. \tag{3.6}$$

As in the present case $\sigma_{ll}^{\varepsilon} = \hat{\sigma}_{ab} v_{a,b}$ (compare with (2.3)), and $\hat{\sigma}_{ab} = E_1 U_{l,l}^{ab}$ (E_I is Young's modulus for the material of beam I), we have

$$\hat{l}_{ij\alpha\beta ab}(N^{\gamma\delta}) = (\mathrm{mes}P_1)^{-1} \sum_{I=1}^{N} \int_L E_I [U_{l,l}^{ab} U_{l,l}^{\alpha\beta} U_{l,l}^{ij} - \gamma_l^{\alpha}\gamma_l^{\beta} U_{l,l}^{ab}(U_{l,l}^{ij} + U_{l,l}^{\alpha\beta}) + \gamma_l^{i}\gamma_l^{j}\gamma_l^{\alpha}\gamma_l^{\beta} U_{l,l}^{ab}] dl dn d\tau. \tag{3.7}$$

The stresses and strains in Eqs. (3.5) are made up of stretching-compression strains and bending ones and take the form A + Bn + Cτ, where A, B, C = const. One calculates the expressions in Eqs. (3.5) by integrating functions of the form $n^k \tau^L$ over the cross section of the beam. The corresponding kinematic hypotheses must be used if the beams are not thin.

3.2. Periodic-Structure Truss. Let the bending stresses and strains in the CC be negligible. Then $\sigma_{ll}^{\varepsilon}$, $U_{d,d}^{ab}$ (d = l, n, τ) in Eqs. (3.5)-(3.7) are constants [15-17], so Eqs. (3.6) and (3.7) become

$$l_{ij\alpha\beta}(\sigma_{mn}^{\varepsilon}, U^{\gamma\delta}) = (\mathrm{mes}P_1)^{-1} \sum_{I=1}^{N} N_I [e_I^{\alpha\beta} e_I^{ij} - \gamma_l^{\alpha}\gamma_l^{\beta}(e_I^{ij} + e_I^{\alpha\beta}) + \gamma_l^{i}\gamma_l^{j}\gamma_l^{\alpha}\gamma_l^{\beta}] l_I; \tag{3.8}$$

$$\hat{l}_{ij\alpha\beta ab}(N^{\gamma\delta}) = (\mathrm{mes}P_1)^{-1} \sum_{I=1}^{N} \mathscr{E}_I [e_I^{ab} e_I^{\alpha\beta} e_I^{ij} - \gamma_l^{\alpha}\gamma_l^{\beta} e_I^{ab}(e_I^{ij} + e_I^{\alpha\beta}) + \gamma_l^{i}\gamma_l^{j}\gamma_l^{\alpha}\gamma_l^{\beta}] l_I, \tag{3.9}$$

in which N_I is the axial force ($\sigma_{ll}^{\varepsilon}$ multiplied by the cross-sectional area of the beam), e_I^{ij} the axial strain, l_I the length, and $\mathscr{E}_I$ the tensional rigidity of rod I.

3.3. Construction Containing Plates. When one considers the plates, the analysis may be performed by analogy with the above, with the stresses and strains determined in accordance with the kinematic hypotheses, after which the integration over the volumes of the plates may be performed explicitly.

3.4. Semimonocoque Constructions. Let the rod framework in a CC take up the tension/compression (see Sec. 3.2), while the plates work only in shear [6]. Then in a coordinate system linked to a plate

$$\left.\begin{array}{l} \sigma_{ab}^{\varepsilon} = \mathrm{const} \\ N_{a,b}^{ij} = \mathrm{const} = \Gamma^{ij} \end{array}\right\} \text{ for } ab = ln,\, nl, \quad \left.\begin{array}{l} \sigma_{ab}^{\varepsilon} = 0 \\ N_{a,b}^{ij} = 0 \end{array}\right\} \text{ for } ab \neq ln,\, nl.$$

As a result, for the plates

$$l_{ij\alpha\beta}^{pl}(\sigma_{mn}^{\varepsilon}, N^{\gamma\delta}) = (\mathrm{mes}P_1)^{-1} \sum_{I=1}^{N} S_I [\Gamma_I^{\alpha\beta}\Gamma_I^{ij} - \Gamma_I^{ij}\gamma_n^{\alpha}\gamma_l^{\alpha} - \Gamma_I^{\alpha\beta}\gamma_n^{i}\gamma_l^{j} + \gamma_a^{i}\gamma_n^{j}\gamma_d^{\alpha}\gamma_l^{\beta} S_I \delta_d^{a}] \mathscr{L}_I,\ a, b = l, n. \tag{3.10}$$

Here S_I are the shearing forces ($\sigma_{nl}^{\varepsilon}$ multiplied by the plate thickness), $\Gamma_I^{\alpha\beta}$ the shearing strain, and $\mathscr{L}_1$ the area of plate I in plan. The summation in Eq. (3.10) is taken over the number of plates in the CC. One should add Eq. (3.10) to Eq. (3.8) to get the final expression for $l_{ij\alpha\beta}$.

4. Methods of Solving the CP.

The most general method of determining $\sigma_{ij}^{\varepsilon}$ and $N^{\alpha\beta}$ is to solve Eq. (2.2) numerically. Examples have been given for instance in [18] for the use of the finite-element method. However, if the CC is formed by thin-walled elements and has complicated geometry, standard numerical methods are ineffective and it is logical to use methods that explicitly incorporate the thin walls of the CC elements. The [12, 13] approach is one such method. The method proposed there involves replacing the CP in elasticity theory by the CP in the theory of beams/plates and agrees with the analysis method for finite-dimensional structures that has been thoroughly developed in the sense of theoretical analysis and in the sense of software. We consider applying the last method to the CP. We introduce the generalized displacements of the CC nodes ($\mathbf{u}_1$, $\mathbf{m}_1$, ...), in which $\mathbf{u}_1$, ... are the displacements proper and $\mathbf{m}_1$, ... are the residual components of the generalized-displacement vector (e.g., the angles of rotation for the ends of the beams and so on). The CP takes the form

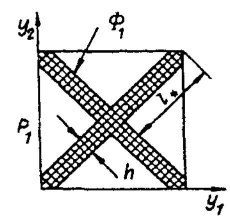

Fig. 2

$$TW^{\alpha\beta} = 0 \text{ at the interior nodes} \tag{4.1}$$

$$(tW^{\alpha\beta})_{a+} = (tW^{\alpha\beta})_{a-} \text{ at the boundary nodes} \tag{4.2}$$

$$(W^{\alpha\beta} - y_\alpha e_\beta)_{a+} = (W^{\alpha\beta} - y_\alpha e_\beta)_{a-} \text{ at the boundary nodes} \tag{4.3}$$

$$\sum_{I=1}^{N} W_I^{\alpha\beta} - y_{\alpha I} e_\beta = \sum_{I=1}^{N} m_I^{\alpha\beta} = 0,$$
$$W^{\alpha\beta} = (u_1^{\alpha\beta}, m_1^{\alpha\beta}, \ldots, u_N^{\alpha\beta}, m_N^{\alpha\beta}). \tag{4.4}$$

Here (4.1) are the equations of equilibrium (T is the influence matrix) [6, 19]; Eqs. (4.2) and (4.3) are the periodicity conditions (the subscripts $a+$ and $a-$ denote those corresponding one to the other at opposite faces of the PC); $tW^{\alpha\beta}$ are the forces at the boundary nodes; Eq. (4.4) is the analog of $\langle N^{\alpha\beta}\rangle = 0$; N the number of CC elements; and $y_\alpha e_\beta$ takes the values at the nodes of the PC. The Eqs. (4.1)-(4.4) treatment is a system of linear algebraic equations that has a unique solution by virtue of condition (4.4).

One solves Eqs. (4.1)-(4.4) to recover $U^{\alpha\beta}$ in the regions L_I occupied by the elements on the basis of the kinematic hypotheses, and one then calculates l_{ijkl} or $l_{ijkl\alpha\beta}$ in accordance with the above formulas. For typical constructional elements such as rods, beams, and plates, one can obtain explicit expressions for l_{ijkl} and $l_{ijkl\alpha\beta}$ in terms of the generalized displacements of the ends.

The following formula applies:

$$a_{ijkl}(0) = \sum_{a \in \Gamma_j} (tW^{kl})_i, \tag{4.5}$$

where the summation is taken over the nodes on face j of the PC.

To calculate l_{ijkl} and $l_{ijkl\alpha\beta}$, which characterize the effects from the initial stresses in a finite-dimensional structure, it is thus effective to use a matrix method [6, 19].

Note 4.1. The matrix method can be applied to the construction as a whole (see for example [6] of incorporating the initial stresses and strains in a structure with finite dimensions). A large linear-equation system arises, which can be analyzed by analogs of the averaging method. When there is any reason to assume that the construction does not permit of an averaged description, one can use direct methods of examining the equation system thereby arising.

5. Examples.

Example 1 (X-shaped PC, Fig. 2). The solution to the CP for the beam CC shown in Fig. 2 can be obtained in an explicit form. We use the summary of the CC and consider one beam (1/4 of the CC). To derive U^{11}, one needs to solve the bending-tension problem for that beam subject to the edge conditions

$$w'(0) = w'(l_*) = v(0) = 0,\ w(l_*) = v(l_*),\ w(l_*) + v(l_*) = 1/\sqrt{2},$$

in which w is the normal deflection and v the axial displacement (in the local coordinate system), with the coordinate l reckoned from zero (the center of the CC) and l_* the beam length.

The classical equations [15] give

$$w = \frac{1}{\sqrt{2}l_*^3}\left(l^3 - \frac{3}{2}l_* l^2\right),\ v = \frac{1}{2}l.$$

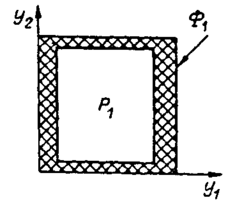

Fig. 3

Some change in the methods given in [12, 12] is needed to link up the elements at the faces of the PC that appear in two faces at once (Fig. 2). In the present case, by virtue of the CC symmetry, the vectors for the normal forces are normal to the faces of the PC and Eq. (4.5) gives

$$a_{1111}(0) = 2(N + Q)/\sqrt{2} = \mathscr{E} + 12D,$$
$$a_{2211}(0) = 2(N - Q)/\sqrt{2} = \mathscr{E} - 12D \tag{5.1}$$

($\mathscr{E}$ and D are the rigidities of the beam in tension and bending, while N and Q are the axial and shearing forces [15]).

The strains and stresses are as follows in the beam considered as a three-dimensional body [15-17] in the $\{l, n, \tau\}$ local coordinate system:

$$N^{11}_{l,l} = v' + nw'' = \frac{1}{2\sqrt{2}} + n\frac{6l - 3l_*}{2l_*^3},$$
$$N^{11}_{n,n} = N^{11}_{\tau\tau} = -\nu N^{11}_{l,l},\ N^{11}_{a,b} = 0 \text{ for } a \neq b, \tag{5.2}$$
$$\sigma^{\varepsilon}_{ll} = EN^{11}_{l,l},\ \sigma^{\varepsilon}_{ab} = 0 \text{ for } ab \neq ll.$$

We then get Eqs. (3.6) and (3.7) containing the $N_{l,l}{}^{\alpha\beta}$ and $\sigma_{l,l}{}^{\varepsilon}$ given by Eqs. (5.2).

Let the averaged stress be $\hat{\sigma}_{11} \neq 0$, $\hat{\sigma}_{ij} = 0$ for ij $\neq$ 11. From Eqs. (5.1) we calculate the averaged strains (i.e., $J_{\alpha\beta mn}\hat{\sigma}_{mn}$):

$$e_{11} = \frac{a_{1111}(0)}{\Delta}\hat{\sigma}_{11},\ e_{22} = \frac{a_{1122}(0)}{\Delta}\hat{\sigma}_{11},$$

where $\Delta = 24\mathscr{E}$); $e_{ij} = 0$ for ij $\neq$ 11.22. The value of 1/4 l_{111111} for 1/4 of the CC (i.e., a beam) is

$$E\int_0^{l_*}\int_{-h/2}^{h/2}[(N^{11}_{l,l})^3 - (N^{11}_{l,l})^2 + \frac{1}{2}N^{11}_{l,l}]dldn.$$

We substitute here for $N_{l,l}{}^{11}$ from Eq. (5.2) and integrate to get for the entire CC that

$$l_{111111} = 4\frac{5 - 2\sqrt{2}}{32}\mathscr{E}l_* = \frac{5 - 2\sqrt{2}}{8}\mathscr{E}l_*.$$

To derive l_{111122}, it is sufficient to note that one can use the CC symmetry and calculate N^{22} by analogy with N^{11} in a coordinate system turned through 90°, so for a beam

$$N^{22}_{l,l} = \frac{1}{2\sqrt{2}} - n\frac{6l - 3l_*}{2l_*^3},$$

and to calculate l_{111122} one should compute the integral

$$E\int_0^{l_*}\int_{-h/2}^{h/2}[N^{22}_{l,l}(N^{11}_{l,l})^2 - N^{22}_{l,l}N^{11}_{l,l} + \frac{1}{2}N^{22}_{l,l}]dldn.$$

Integration gives

$$l_{111122} = \frac{5 - 2\sqrt{2}}{8} \mathscr{E}.$$

Finally, for this case we get

$$\begin{aligned} a_{1111}(\sigma) &= \mathscr{E} + 12D + \left[\frac{5 - 2\sqrt{2}}{8} \frac{\mathscr{E} + 12D}{24D} + 2h\sqrt{2}\right]\dot{\sigma}_{11}, \\ a_{1122}(\sigma) &= \mathscr{E} - 12D + \frac{5 - 2\sqrt{2}}{8} \frac{\mathscr{E} + 12D}{24D} \dot{\sigma}_{11} = a_{2211}(\sigma), \\ a_{2222}(\sigma) &= \mathscr{E} + 12D + \frac{5 - 2\sqrt{2}}{8} \frac{\mathscr{E} + 12D}{24D} \dot{\sigma}_{11}. \end{aligned} \tag{5.3}$$

The last equation follows from the CC symmetry.

Intermediate averaging will give the following values for the quantities from Eqs. (5.3):

$$\mathscr{E} + 12D + \dot{\sigma}_{11}, \mathscr{E} - 12D, \mathscr{E} + 12D. \tag{5.4}$$

The discrepancies between Eqs. (5.3) and (5.4) are of the order of D (D + h for $\sigma_{1111}(\sigma)$), because the bending strains are responsible for the differences between the strains corresponding to $U^{\alpha\beta}$ (i.e., the local strains) and the homogeneous strains corresponding to the displacement $y_{\alpha e_\beta}$.

Example 2 (rectangular PC, Fig. 3). Let the averaged load be directed along the axis (e.g., the weight). Then with $\alpha\beta = 11.22$ we have $\mathbf{U}^{\alpha\beta} = y_\alpha{}^e\mathbf{u}_\beta$, and so $l_{ijkl}(\sigma_{mn}{}^\varepsilon, \mathbf{N}^{\alpha\beta}) = 0$.

Then the difference between $l_{ijkl}(\sigma_{mn}{}^\varepsilon, \mathbf{N}^{\alpha\beta})$ and zero is due to discrepancies between the local strains defined from the CP solution and the homogeneous strains corresponding to the global ones.

6. Application.

6.1. Vibrations in Stressed Structures. The equations for the natural oscillations are derived from (1.2) by replacing $\rho\mathbf{u}_{tt}{}^\varepsilon$ by $\lambda_\varepsilon{}^2\rho\mathbf{u}^\varepsilon$ for the initial construction and $\langle\rho\rangle\mathbf{u}_{,tt}$ by $\lambda^2\langle\rho\rangle\mathbf{u}$ for the averaged one. Here λ_ε and λ are the natural frequencies for the real and averaged constructions. The finite sets of eigenvalues in a frame converge [13, 14] for $\varepsilon \to 0$ when one incorporates the multiplicities of the initial and averaged treatments (see [20] for details) provided that the solution to the initial and averaged treatments in Eqs. (1.2) and (1.4) converge. The basis may be given by analogy with [21-24].

6.2. Planar Waves. The initial problem does not allow solution as plane waves in the general case, while such solutions exist for the averaged problem. By virtue of (1.4), the plane waves are described by $a_{ijkl}(\sigma)n_jn_lX_k = c^2\langle\rho\rangle X_i$.

Note 6.1. It is assumed above that the operators $L_\varepsilon(\sigma)$ and $\hat{L}(\sigma)$ have compact inverses defined on all $H^1(Q_\varepsilon)$ and $H^1(Q)$ correspondingly. That condition may be violated. In particular, there is a question of stability loss in a stressed construction with respect to averaged forms. One can say in advance that intermediate averaging will be inapplicable in that case.

Note 6.2. The initial stresses for a thin body can sometimes be averaged within the framework of beam theory. The decisive part is played by the order of the initial stresses by comparison with the beam diameter [25, 26].

7. Conclusions.

1. The averaging method should be applied directly to the initial treatment in considering stressed three-dimensional constructions.

2. The working formulas derived here for initial stresses small by comparison with the elastic constants are applicable for most artificial structures.

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